Section 3.3: Linear programming: A geometric approach

In addition to constraints, linear programming problems usually involve some quantity to maximize or minimize such as profits or costs. The quantity to be maximized or minimized translates to some linear combinations of the variables called an objective function. These problems involve choosing a solution from the feasible set for the constraints which gives an optimum value (maximum or a minimum) for the objective function.

Example A juice stand sells two types of fresh juice in 12 oz cups. The Refresher and the Super Duper. The Refresher is made from 3 oranges, 2 apples and a slice of ginger. The Super Duper is made from one slice of watermelon and 3 apples and one orange. The owners of the juice stand have 50 oranges, 40 apples, 10 slices of watermelon and 15 slices of ginger. Let x denote the number of Refreshers they make and let y denote the number of Super Dupers they make.

Last day, we saw that the **set of constraints** on x and y were of the form :

$$3x + y \leq 50$$

$$2x + 3y \leq 40$$

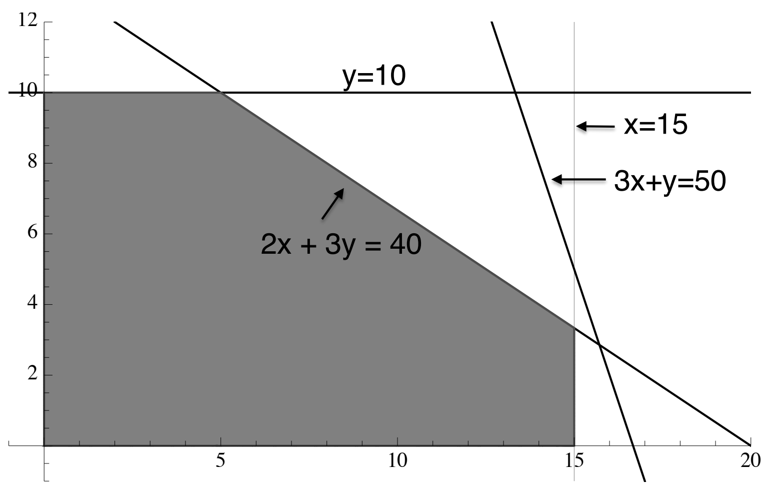
$$x \leq 15$$

$$y \leq 10$$

$$x \geq 0$$

$$y \geq 0$$

The conditions $x \geq 0$ and $y \geq 0$ are called **non-negative conditions.** We can now draw the feasible set to see what combinations of x and y are possible given the limited supply of ingredients:



Now suppose that the Refreshers sell for \$6 each and the Super Dupers sell for \$8 each. Lets suppose also that the juice stand will sell all of the drinks they can make on this day, then their revenue for the day is $6x + 8y$. Lets assume also that one of the goals of the juice stand is to maximize revenue. Thus they want to maximize the value of $6x + 8y$ given the constraints on production listed above. In other words they want to find a point (x, y) in the feasible set which gives a maximum value for the **objective function** $6x + 8y$. [Note that the value of the objective function ($6x + 8y =$ revenue) varies as (x, y) varies over the points in the feasible set. For example if $(x, y) = (2, 5)$, revenue $= 6(2) + 8(5) = \$52$, whereas if $(x, y) = (5, 10)$, revenue $= 6(5) + 8(10) = \$110$.]

Terminology: A linear inequality of the form

$$a_1x + a_2y \le b, \quad a_1x + a_2y < b, \quad a_1x + a_2y \ge b, \text{ or } a_1x + a_2y > b,$$

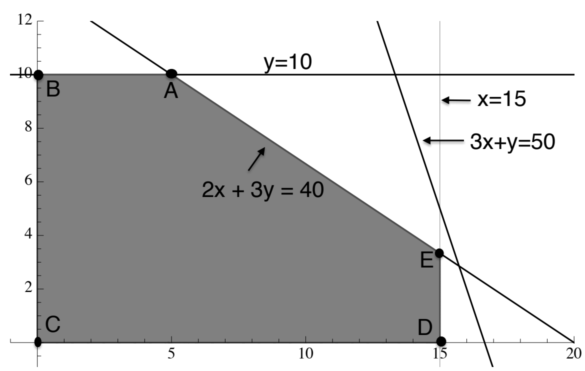
where a_1, a_2 and b are constants, is called a **constraint** in a linear programming problem. The restrictions $x \ge 0$, $y \ge 0$ are called **nonnegative conditions**. A **linear objective function** is an expression of the form $cx + dy$, where c and d are constants, for which one needs to find a maximum or minimum on the feasible set. The term **optimal value** refers to the sought after maximum or minimum as the case may be.

Theorem Given a linear objective function subject to constraints in the form of linear inequalities, **if the objective function has an optimal value (maximum or minimum) on the feasible set**, it must occur at a corner(or vertex) of the feasible set
(as long as the F.S. has vertices, if the F.S. has no vertices the max. and min. when they exist occur on the boundary lines (which must be parallel if there are no vertices)).

Example from above We can summarize our problem from the previous example in the following form:
Find the maximum of the objective function $6x + 8y$ subject to the constraints

$$\begin{array}{cc} 3x + y \le 50, & 2x + 3y \le 40, \\ x \le 15, & y \le 10, \\ x \ge 0, & y \ge 0. \end{array}$$

From the above theorem, we know that the maximum of $6x + 8y$ on the feasible set occurs at a corner of the feasible set (it may occur at more than one corner, but it occurs at at least one). We already have a picture of the feasible set and below, we have labelled the corners, A, B, C, D and E.



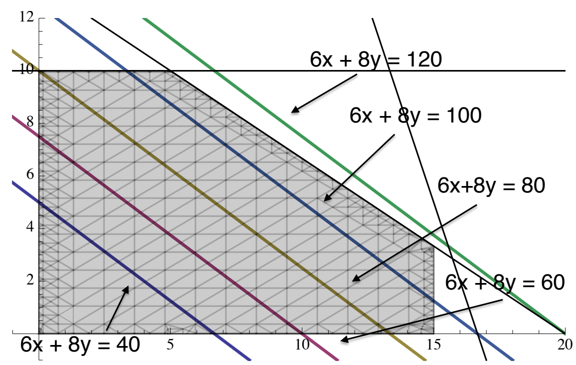
To find the maximum value of $6x + 8y$ and a point (x, y) in the feasible set at which it is achieved, we need only calculate the co-ordinates of the points A, B, C, D and E and compare the value of $6x + 8y$ at each.

Point	Coordinates	Value of $6x + 8y$
A		
B		
C	(0, 0)	0
D		
E		

Method for finding optimal values: Given a set of linear constraints and a linear objective function, to find the optimal value of the objective function subject to the given constraints;

- Draw the feasible set and identify which lines form the boundaries.
- Calculate the coordinates of the corners by finding the intersection of the relevant lines.
- Evaluate the objective function at each corner.
- Determine which corners give the optimal value (maximum or minimum) of the objective function.

To get some idea of why the optimal values occur at the corners, look at the picture below. It shows lines of the form $6x+8y=c$ where c takes the values $40, 60, 80, 100$ and 120. The value of the objective function increases along a line perpendicular to those drawn, so the maximum value of the objective function will appear somewhere along the boundary of the feasible set. The set of points at which the maximum is achieved will always include a vertex (in our case it is limited to a vertex) but may include a portion of a line along the boundary.



Given the assumptions of the above theorem,
if the feasible set is bounded, a maximum and a minimum value of the objective function will exist among the values at the corner points.
(Note: if the boundary lines are not in the F.S., then the vertex in question may not be feasible, but one can get close to the max./min. by choosing points in the F.S. close to this vertex.
If the feasible set is unbounded an optimal value may not exist, but if it does, it will occur at a corner point, (unless the F.S. has no vertices (boundary is parallel lines) in which case the max or min must occur on the boundary if it exists.) If the objective function is of the form $cx+dy$ where c and d are > 0 and the constraints have nonnegative conditions, then the objective function will achieve a minimum value at a corner of the unbounded feasible set, but will have no maximum.

Example Mr. Carter eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Carter's breakfast should provide at least 480 calories but less than 700 milligrams of sodium. Mr. Carter would like to maximize the amount of protein in his breakfast mix.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

Let x denote the number of ounces of Cereal A that Mr. Carter has for breakfast and let y denote the number of ounces of Cereal B that Mr. Carter has for breakfast. In the last lecture, we found that the set of constraints for x and y were

$$100x + 140y \geq 480, \qquad 150x + 190y \leq 700, \qquad x \geq 0, \quad y \geq 0$$

(a) What is the objective function?

(b) Graph the feasible set. (Note: Once can multiply or divide an inequality by a positive number without changing the solution set just as one would an equation. [When we multiply or divide an inequality by a negative number you must reverse the inequality])

(c) Find the vertices of the feasible set and the maximum of the objective function on the feasible set.

Example Michael is taking a timed exam in order to become a volunteer firefighter. The exam has 10 essay questions and 50 Multiple choice questions. He has 90 minutes to take the exam and knows he cannot possibly answer every question. The essay questions are worth 20 points each and the short-answer questions are worth 5 points each. An essay question takes 10 minutes to answer and a shot-answer question takes 2 minutes. Michael must do at least 3 essay questions and at least 10 short-answer questions. Michael knows the material well enough to get full points on all questions he attempts and wants to maximize the number of points he will get. Let x denote the number of multiple choice questions that Michael will attempt and let y denote the number of essay questions that Michael will attempt. Write down the constraints and objective function in terms of x and y and find the/a combination of x and y which will allow Michael to gain the maximum number of points possible.

Example with unbounded region A local politician is budgeting for her media campaign. She will distribute her funds between TV ads and radio ads. She has been given the following advice by her campaign advisors;

- She should run at least 120 TV ads and at least 30 radio ads.
- The number of TV ads she runs should be at least twice the number of radio ads she runs but not more than three times the number of radio ads she runs.

The cost of a TV ad is \$8000 and the cost of a radio ad is \$2000. Which combination of TV and radio ads should she choose to minimize the cost of her media campaign?

Example: No Feasible region Mr. Baker eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Baker's breakfast should provide at least 600 calories but less than 700 milligrams of sodium. Mr. Baker would like to maximize the amount of protein in his breakfast mix.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

Let x denote the number of ounces of Cereal A that Mr. Baker has for breakfast and let y denote the number of ounces of Cereal B that Mr. Baker has for breakfast.

(a) Show that the feasible set is empty here, so that there is no feasible combination of the cereals for Mr. Baker's breakfast.

(b) If Mr. Baker goes shopping for new cereals, what should he look for on the chart giving the Nutritional value, so that he can have some feasible combination of the cereals for breakfast?